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Existence, Stability, and Convergence of Solutions of Discrete Velocity Models to the Boltzmann Equation

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We prove the convergence of finite-difference approximations to solutions of the Boltzmann equation. An essential step is the proof of convergence of discrete approximations to the collision integral. This proof relies on our previous results on the consistency of this approximation. For the space-homogeneous problem we prove strong convergence of our discrete approximation to the strong solution of the Boltzmann equation. In the space-dependent case we prove weak convergence to DiPerna–Lions solutions.

KEY WORDS: Boltzmann-equation; discrete velocity models; convergence of discrete approximation; kinetic theory; numerical methods.

1. INTRODUCTION

The problem of numerical approximation of the Boltzmann equation has a long tradition and a huge literature. Recently a particular interest has been focused on methods which can be considered as finite difference approximations.[3, 13] In our previous paper[15] we have discussed one of such algorithms introduced by Goldstein *et al.*[9] (cf. also other papers devoted to this subject.[2, 10, 11]) We have shown the consistency of the algorithm and made an error estimation. In the present paper we are addressing the problem of convergence of this algorithm.

To prove that solutions of a certain numerical approximation are convergent to a solution of the corresponding continuous problem we have to

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know that the continuous problem possesses a solution. In the case of the Boltzmann equation, we know that for general initial data a solution exists only in a weak sense. That cause additional troubles in the proof of convergence. Hence we decided to consider separately two cases. The space homogeneous equation, where there exists the unique strong solution of the corresponding Cauchy problem. In that case, we are able to prove strong convergence of the numerical algorithm. For the space dependent Boltzmann equation, we have to approximate the weak solutions of DiPerna–Lions. This problem has been partially solved by Mischler.[13] We use his results and our proof of convergence for the collision integral[15] to show a weak convergence of a certain approximation to the DiPerna–Lions solution.

2. NOTATION AND PRELIMINARY RESULTS

Let us consider the Cauchy problem for the space-homogeneous Boltzmann equation

$$\begin{aligned} \frac{\partial f}{\partial t} &= Q(f, f) \\ f(0) &= f^0 \end{aligned} \tag{2.1}$$

with the collision term $Q(f, f)$ defined by the formula

$$Q(f, f) = \int_{\mathbb{R}^d} \int_{S^{d-1}} d\mathbf{v}_1 \, d\mathbf{u} \, q(|\mathbf{w}|, \mathbf{u})(f(\mathbf{v}')\, f(\mathbf{v}'_1) - f(\mathbf{v})\, f(\mathbf{v}_1)) \tag{2.2}$$

Here $\mathbf{v}$ and $\mathbf{v}_1$ are precollision velocities and $\mathbf{w} = \mathbf{v} - \mathbf{v}_1$. $\mathbf{v}'$ and $\mathbf{v}'_1$ are postcollision velocities given by the expressions

$$\begin{aligned} \mathbf{v}' &= \frac{\mathbf{v} + \mathbf{v}_1}{2} + \mathbf{u}\, \frac{|\mathbf{v} - \mathbf{v}_1|}{2} \\ \mathbf{v}'_1 &= \frac{\mathbf{v} + \mathbf{v}_1}{2} - \mathbf{u}\, \frac{|\mathbf{v} - \mathbf{v}_1|}{2} \end{aligned} \tag{2.3}$$

In formula (2.2) $d\mathbf{u}$ denotes the integration with respect to the normalized Lebesgue measure on the surface of the unit sphere S^{d-1}, i.e., $\mu(S^{d-1}) = 1$, and $q(w, \mathbf{u})$ is the collision kernel which is assumed to fulfill the Carleman condition

$$0 \leqslant q(w, \mathbf{u}) \leqslant c(1 + w) \tag{2.4}$$

This equation is approximated by a Discrete Velocity Model (DVM) on a discrete lattice in the velocity space. To this end, we introduce in $\mathbb{R}^d$ a regular grid $\Omega_h = \{\mathbf{v} \in \mathbb{R}^d : \mathbf{v} = h\mathbf{n},\ \mathbf{n} \in \mathbb{Z}^d\}$, where h is an arbitrary positive number (mesh step). Let $f_i(t) = f(\mathbf{v}_i, t)$, where $\mathbf{v}_i \in \Omega_h$, and f_D denote the sequence $\{f_i\}$. Then a discrete version of (2.1) can be written as

$$\begin{aligned} \frac{\partial f_i}{\partial t} &= \sum_{j,k,l \in I_i} \Gamma_{ij}^{kl}(f_k f_l - f_i f_j) \\ f_i(0) &= f_i^0 \end{aligned} \tag{2.5}$$

The summation in this equation is in fact taken over a set of grid points, i.e., indices i, j, k and l are abbreviations for $\mathbf{v}_i$, $\mathbf{v}_j$, $\mathbf{v}_k$ and $\mathbf{v}_l$ and the set I_i is defined as follows. For two incoming velocities $\mathbf{v}_i$ and $\mathbf{v}_j$ we set

$$V_{ij} = \{\mathbf{v}_k, \mathbf{v}_l \in \Omega_h : \mathbf{v}_k + \mathbf{v}_l = \mathbf{v}_i + \mathbf{v}_j,\ \mathbf{v}_k^2 + \mathbf{v}_l^2 = \mathbf{v}_i^2 + \mathbf{v}_j^2\}$$

with r_{ij} being the cardinality of V_{ij}. Then

$$I_i = \bigcup_{\mathbf{v}_j \in \Omega_h} \{\mathbf{v}_j\} \times V_{ij}$$

The coefficients Γ_{ij}^{kl} are approximations to the continuous collision kernel in integral (2.2)

$$\Gamma_{ij}^{kl} = \frac{1}{r_{ij}} h^d q_{ij}^{kl}$$

where $q_{ij}^{kl} = q(|\mathbf{v}_i - \mathbf{v}_j|, (\mathbf{v}_k - \mathbf{v}_l)/|\mathbf{v}_k - \mathbf{v}_l|)$.

That approximation was discussed in details in our previous papers.[2, 15] Using the classical theory of DVM,[4, 8] it is easy to prove that this model possesses essential features of the full Boltzmann equation: conservation laws, H-theorem and existence of stationary states in the form of Gaussian distributions. It can also be shown that the space of summational invariants is reduced to mass, momentum and energy. For more details we refer to other publications.[3, 12, 16]

To make a comparison between solutions of (2.1) and (2.5) we have to define discrete solutions on the whole $\mathbb{R}^d$. To this end, let us observe that the grid Ω_h defines a partition of $\mathbb{R}^d$ into a countable set of cells Λ_i of size h^d and centers in points $\mathbf{v}_i = h\mathbf{n}$ (like previously with velocities, we use for cells a single index, i.e., writing Λ_i for a cell with center $\mathbf{v}_i$). Then we can extend a discrete solution on the whole $\mathbb{R}^d$ as a step function constant on cells Λ_i. An essential difficulty in that procedure is the correct definition of the collision kernel.

First, let us replace kernel $q(w, \mathbf{u})$ with the $4d$-dimensional symmetric collision kernel

$$\sigma(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}_1') = q\left(|\mathbf{v}-\mathbf{v}_1|, \frac{\mathbf{v}'-\mathbf{v}_1'}{|\mathbf{v}'-\mathbf{v}_1'|}\right) \delta(\mathbf{v}+\mathbf{v}_1-\mathbf{v}'-\mathbf{v}_1')\, \delta(\mathbf{v}^2+\mathbf{v}_1^2-\mathbf{v}'^2-\mathbf{v}_1'^2)$$

where $\delta(x)$ denotes the Dirac function (in the collision integral with kernel $\sigma(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}_1')$ integration is carried over $\mathbb{R}^{3d}$). To discretize this kernel we proceed as follows. First, we say that a pair of cells (Λ_k, Λ_l) is admissible for a pair (Λ_i, Λ_j), if $\mathbf{v}_k, \mathbf{v}_l \in V_{ij}$, where $\mathbf{v}_\alpha$ denotes the center of a cell Λ_α. Then we can write

$$\sigma_{\mathrm{h}}(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}_1') = \begin{cases} q_{ij}^{kl} & \text{if } \ \mathbf{v} \in \Lambda_i, \mathbf{v}_1 \in \Lambda_j, \mathbf{v}' \in \Lambda_k, \mathbf{v}_1' \in \Lambda_l \\ & \text{and } \ (\Lambda_k, \Lambda_l) \text{ is admissible for } (\Lambda_i, \Lambda_j) \\ 0 & \text{otherwise} \end{cases}$$

Defining the step function

$$C_{\mathrm{h}}(\mathbf{v}) = \mathbf{v}_i, \qquad \text{if} \quad \mathbf{v} \in \Lambda_i$$

we can write

$$\sigma_{\mathrm{h}}(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}_1') = \sigma(C_{\mathrm{h}}(\mathbf{v}), C_{\mathrm{h}}(\mathbf{v}_1), C_{\mathrm{h}}(\mathbf{v}'), C_{\mathrm{h}}(\mathbf{v}_1'))$$

σ_{h} is therefore the modified collision kernel as proposed by Mischler.[13]

Let now f_{h} be a step function corresponding to the solution $f_D = \{f_i\}$ of DVM in the whole $\mathbb{R}^d$, i.e.,

$$f_{\mathrm{h}}(t, \mathbf{v}) = f_i(t), \qquad \text{for} \quad \mathbf{v} \in \Lambda_i$$

Then f_{h} is a solution of the Cauchy problem

$$\begin{gathered} \frac{\partial f_{\mathrm{h}}}{\partial t} = Q_{\mathrm{h}}(f_{\mathrm{h}}, f_{\mathrm{h}}) \\ f_{\mathrm{h}}(0, \mathbf{v}) = f_D^0(\mathbf{v}) = f^0(C_{\mathrm{h}}(\mathbf{v})) \end{gathered} \tag{2.6}$$

where Q_{h} denotes the collision integral with kernel σ_{h} and an obvious extension of integration domain to $\mathbb{R}^{3d}$.

We also consider truncated versions of (2.1) and (2.5) introducing collision kernels with compact support[12]

$$\sigma^n(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}'_1) = \begin{cases} \sigma(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}'_1) & \text{if} \quad \mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}'_1 \in B(0, n) \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

and its corresponding discrete version

$$\sigma^n_{\mathrm{h}}(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}'_1) = \begin{cases} \sigma_{\mathrm{h}} & \text{if} \quad C_{\mathrm{h}}(\mathbf{v}), C_{\mathrm{h}}(\mathbf{v}_1), C_{\mathrm{h}}(\mathbf{v}'), C_{\mathrm{h}}(\mathbf{v}'_1) \in B(0, n) \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

Distribution functions f and f_{h} truncated to the ball $B(0, n)$ are denoted by f^n and f^n_{h}, respectively. They are solutions to the Cauchy problems:

$$\begin{aligned} &\frac{\partial f^n}{\partial t} = Q^n(f^n, f^n) \\ &f^n(0, \mathbf{v}) = \begin{cases} f^0(\mathbf{v}) & \text{if} \quad |\mathbf{v}| \leqslant n \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\frac{\partial f^n_{\mathrm{h}}}{\partial t} = Q^n_{\mathrm{h}}(f^n_{\mathrm{h}}, f^n_{\mathrm{h}}) \\ &f^n_{\mathrm{h}}(0, \mathbf{v}) = \begin{cases} f^0(C_{\mathrm{h}}(\mathbf{v})) & \text{if} \quad |C_{\mathrm{h}}\mathbf{v}| \leqslant n \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{2.10}$$

Here Q^n and Q^n_{h} are collision operators corresponding to collision kernels σ^n and σ^n_{h}, respectively.

The functions f, f^n, f_{h} and f^n_{h}, which are solutions to the Cauchy problems (2.1), (2.9), (2.6) and (2.10), will be considered in the weighted L^1-spaces

$$L^1_s = \left\{ f \colon \int_{\mathbb{R}^g} |f(\mathbf{v})| \, (1 + \mathbf{v}^2)^s \, d\mathbf{v} < +\infty \right\}$$

with the norm

$$\|f\|_s = \int_{\mathbb{R}^d} |f(\mathbf{v})| \, (1 + \mathbf{v}^2)^s \, d\mathbf{v}$$

f_D which is a solution of DVM (2.5) will be considered in l^1-type spaces with polynomial weight

$$l^1_s = \left\{ f_D \colon h^d \sum_{\mathbf{v}_i \in \Omega_h} |f_i| \, (1 + \mathbf{v}_i^2)^s < +\infty \right\}$$

and the norm

$$\|f_D\|_s = h^d \sum_{\mathbf{v}_i \in \Omega_h} |f_i| \, (1+\mathbf{v}_i^2)^s$$

Remark 1. In what follows, integration of step functions in $\mathbb{R}^d$ can be treated in two different ways. First, integration can be considered as performed with respect to the usual Lebesgue measure in $\mathbb{R}^d$. This is the way we are integrating in all estimations calculated in the paper. Second, we can think of integration with respect to an atomic measure concentrated on the grid points $\mathbf{v}_i$, with the measure of each point equal to h^d. This is the way we are integrating in all calculations that lead to conservation laws.

We end this section by proving some useful properties of the introduced models. First, we show that classical symmetry properties of the collision kernel σ still hold for the kernels σ^n, σ_{h} and σ_{h}^n.

Lemma 1. Let σ be one of the collision kernels σ^n, σ_{h} or σ_{h}^n. Then

(i) $\sigma(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}_1') = \sigma(\mathbf{v}_1, \mathbf{v}, \mathbf{v}_1', \mathbf{v}')$,

(ii) $\sigma(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}_1') = \sigma(\mathbf{v}', \mathbf{v}_1', \mathbf{v}, \mathbf{v}_1)$,

(iii) $\int_{\mathbb{R}^{2d}} \sigma(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}_1') \, d\mathbf{v}' \, d\mathbf{v}_1' \leqslant c(1+\mathbf{v}^2)^{1/2} \, (1+\mathbf{v}_1^2)^{1/2}$.

Proof. (i) and (ii) are a straightforward consequence of the definition of σ^n, σ_{h} and σ_{h}^n of the symmetry of σ. (iii) is obvious for σ^n since $\sigma^n \leqslant \sigma$.

Let us prove (iii) for $\sigma = \sigma_{\mathrm{h}}$ (the case $\sigma = \sigma_{\mathrm{h}}^n$ is included in that case since $\sigma_{\mathrm{h}}^n \leqslant \sigma_{\mathrm{h}}$). Let $\mathbf{v}_i = C_{\mathrm{h}}(\mathbf{v})$, $\mathbf{v}_j = C_{\mathrm{h}}(\mathbf{v}_1)$ and

$$E = \int_{\mathbb{R}^{2d}} \sigma(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}_1') \, d\mathbf{v}' \, d\mathbf{v}_1' = \frac{1}{r_{ij}} \sum_{k,l} q_{ij}^{kl}$$

where the summation is taken over all grid points on the sphere spanned by $\mathbf{v}_i$ and $\mathbf{v}_j$. By simple estimates we obtain

$$\begin{aligned} E &\leqslant \sup_{\mathbf{u} \in S^{d-1}} q(|\mathbf{v}_i - \mathbf{v}_j|, \mathbf{u}) \leqslant c(1+|\mathbf{v}_i - \mathbf{v}_j|) \\ &\leqslant c(1+|\mathbf{v}+\mathbf{v}_1|) \leqslant c(1+\mathbf{v}^2)^{1/2} \, (1+\mathbf{v}_1^2)^{1/2} \quad \blacksquare \end{aligned}$$

Lemma 2. Let Ω_h be a given discretization of $\mathbb{R}^d$ and σ: $\mathbb{R}^{4d} \to \mathbb{R}^+$ be a non-negative function which satisfies conditions of Lemma 1. Let us assume that for $\sigma(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}'_1) \neq 0$ the following conservation laws are fulfilled

$$\begin{aligned} C_{\mathrm{h}}(\mathbf{v}) + C_{\mathrm{h}}(\mathbf{v}_1) &= C_{\mathrm{h}}(\mathbf{v}') + C_{\mathrm{h}}(\mathbf{v}'_1) \\ C^2_{\mathrm{h}}(\mathbf{v}) + C^2_{\mathrm{h}}(\mathbf{v}_1) &= C^2_{\mathrm{h}}(\mathbf{v}') + C^2_{\mathrm{h}}(\mathbf{v}'_1) \end{aligned} \tag{2.11}$$

Let Q_σ denote the collision integral with kernel σ and f, $g \in L^1_{s+1/2}(\mathbb{R}^d)$ be two nonnegative functions. Then

$$\int_{\mathbb{R}^d} (1+\mathbf{v}^2)^s (Q_\sigma(f, f) - Q_\sigma(g, g))\, d\mathbf{v} \leqslant c\, \|f-g\|_{s+1/2}\, \|f+g\|_{s+1/2}$$

where c is independent of h, for bounded h, and the dependence on σ is only through a constant in Carleman's condition (2.4).

Proof.

$$\begin{aligned} &\int_{\mathbb{R}^d} (1+\mathbf{v}^2)^s (Q_\sigma(f, f) - Q_\sigma(g, g))\, d\mathbf{v} \\ &\quad \leqslant \int_{\mathbb{R}^{4d}} (1+\mathbf{v}^2)^s \sigma(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}'_1)\, |f'f'_1 - g'g'_1|\, d\mathbf{v}\, d\mathbf{v}_1\, d\mathbf{v}'\, d\mathbf{v}'_1 \\ &\quad \leqslant \int_{\mathbb{R}^{4d}} (1+\mathbf{v}^2)^s \sigma(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}'_1)\, |ff_1 - gg_1|\, d\mathbf{v}\, d\mathbf{v}_1\, d\mathbf{v}'\, d\mathbf{v}'_1 \end{aligned}$$

Using Lemma 1(iii), we obtain for the second term

$$\begin{aligned} &\int_{\mathbb{R}^{4d}} (1+\mathbf{v}^2)^s \sigma(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}'_1)\, |ff_1 - gg_1|\, d\mathbf{v}\, d\mathbf{v}_1\, d\mathbf{v}'\, d\mathbf{v}'_1 \\ &\quad \leqslant c \int_{\mathbb{R}^{2d}} (1+\mathbf{v}^2)^{s+1/2} (1+\mathbf{v}_1^2)^{s+1/2}\, |ff_1 - gg_1|\, d\mathbf{v}\, d\mathbf{v}_1 \\ &\quad \leqslant c\, \|f-g\|_{s+1/2}\, \|f+g\|_{s+1/2} \end{aligned}$$

For the first term one has to estimate the difference $\mathbf{v}^2 + \mathbf{v}_1^2 - \mathbf{v}'^2 - \mathbf{v}_1'^2$. By the assumptions of the lemma and the estimation $|C_{\mathrm{h}}(\mathbf{v}) - \mathbf{v}| \leqslant \sqrt{d}/2h$, we obtain

$$\mathbf{v}'^2 + \mathbf{v}_1'^2 - \mathbf{v}^2 - \mathbf{v}_1^2 \leqslant ch(|C_{\mathrm{h}}(\mathbf{v})| + |C_{\mathrm{h}}(\mathbf{v}_1)|) + O(h^2) \leqslant ch(|\mathbf{v}| + |\mathbf{v}_1|) + O(h^2)$$

Therefore

$$\mathbf{v}'^2 + \mathbf{v}_1'^2 \leqslant \mathbf{v}^2 + \mathbf{v}_1^2 + ch(1+\mathbf{v}^2)^{1/2} (1+\mathbf{v}_1^2)^{1/2} + O(h^2) \tag{2.12}$$

which leads to the estimation

$$1+\mathbf{v}^2 \leqslant (1+\mathbf{v}'^2)(1+\mathbf{v}_1'^2) + ch(1+\mathbf{v}'^2)^{1/2}(1+\mathbf{v}_1'^2)^{1/2} + O(h^2)$$
$$\leqslant c(1+\mathbf{v}'^2)(1+\mathbf{v}_1'^2)$$

for bounded h. This gives for the first term a bound similar to that obtained for the second term. ∎

A drawback of the last lemma is that one controls variation of moments of Q by higher moments of the difference of arguments. The following lemma due to DiBlasio[6] does not exhibit this drawback. We present here a generalization of that lemma on the discrete case (which includes the continuous one by setting $h=0$).

Lemma 3. Under the assumptions of Lemma 2, for any $f, g \in L^1_{3/2}(\mathbb{R}^d)$ (f, g nonnegative), we have

$$\int_{\mathbb{R}^d} (1+\mathbf{v}^2)\,\operatorname{sgn}(f-g)(Q_\sigma(f,f) - Q_\sigma(g,g))\,d\mathbf{v} \leqslant c\,\|f-g\|_1\,\|f+g\|_{3/2}$$

where c is independent of h, for bounded h, and the dependence on σ is only through a constant in Carleman's condition (2.4).

Proof.

$$\begin{aligned} &Q_\sigma(f,f) - Q_\sigma(g,g) \\ &\quad = \frac{1}{2}\int_{\mathbb{R}^{3d}} ((f'-g')(f_1'+g_1') - (f-g)(f_1+g_1) \\ &\qquad + (f_1'-g_1')(f'+g') - (f_1-g_1)(f+g))\,\sigma(\mathbf{v},\mathbf{v}_1,\mathbf{v}',\mathbf{v}_1')\,d\mathbf{v}_1\,d\mathbf{v}'\,d\mathbf{v}_1' \end{aligned}$$

Hence

$$\begin{aligned} &\int_{\mathbb{R}^d} (1+\mathbf{v}^2)\,\operatorname{sgn}(f-g)(Q_\sigma(f,f) - Q_\sigma(g,g))\,d\mathbf{v} \\ &\quad \leqslant \frac{1}{2}\int_{\mathbb{R}^{4d}} (1+\mathbf{v}^2)(|f'-g'|\,(f_1'+g_1') + |f_1'-g_1'|\,(f'+g') \\ &\qquad + |f_1-g_1|\,(f+g) - |f-g|\,(f_1+g_1))\,\sigma(\mathbf{V})\,d\mathbf{V} \end{aligned}$$

where $\mathbf{V} = (\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}_1')$. To estimate the last integral let us take a nonnegative function ϕ: $\mathbb{R}^{2d} \to \mathbb{R}^+$ and consider the integral

$$J = \int_{\mathbb{R}^{4d}} (1+\mathbf{v}^2)\,\sigma(\mathbf{V})\,\phi(\mathbf{v}',\mathbf{v}_1')\,d\mathbf{V}$$

By conditions (i) and (ii) of Lemma 1 we obtain

$$J=\frac{1}{2}\int_{\mathbb{R}^{4d}}(2+\mathbf{v}^2+\mathbf{v}_1^2)\,\sigma(\mathbf{V})\,\phi(\mathbf{v}',\mathbf{v}_1')\,d\mathbf{V}$$

$$=\frac{1}{2}\int_{\mathbb{R}^{4d}}(2+\mathbf{v}'^2+\mathbf{v}_1'^2)\,\sigma(\mathbf{V})\,\phi(\mathbf{v},\mathbf{v}_1)\,d\mathbf{V}$$

which together with estimation (2.12) gives

$$J\leqslant\frac{1}{2}\int_{\mathbb{R}^{4d}}(1+\mathbf{v}^2)\,\sigma(\mathbf{V})(\phi(\mathbf{v},\mathbf{v}_1)+\phi(\mathbf{v}_1,\mathbf{v}))\,d\mathbf{V}$$

$$+c\int_{\mathbb{R}^{4d}}(h(1+\mathbf{v}^2)^{1/2}\,(1+\mathbf{v}_1^2)^{1/2}+h^2)\,\sigma(\mathbf{V})\,\phi(\mathbf{v},\mathbf{v}_1)\,d\mathbf{V}$$

Let us set $\phi(x,y)=(f(x)-g(x))(f(y)+g(y))+(f(y)-g(y))\times(f(x)+g(x))$.

Then we obtain

$$\int_{\mathbb{R}^d}(1+\mathbf{v}^2)\,\operatorname{sgn}(f-g)(Q_\sigma(f,f)-Q_\sigma(g,g))\,d\mathbf{v}$$

$$\leqslant\int_{\mathbb{R}^{4d}}(1+\mathbf{v}^2)\,\sigma(\mathbf{V})\,|f_1-g_1|\,(f+g)\,d\mathbf{V}$$

$$+ch\int_{\mathbb{R}^{4d}}(1+\mathbf{v}^2)^{1/2}\,(1+\mathbf{v}_1^2)^{1/2}\,\sigma(\mathbf{V})\,|f-g|\,(f_1+g_1)\,d\mathbf{V}$$

$$+ch^2\int_{\mathbb{R}^{4d}}\sigma(\mathbf{V})\,|f-g|\,(f_1+g_1)\,d\mathbf{V}$$

Then using condition (iii) of Lemma 1 we obtain

$$\int_{\mathbb{R}^d}(1+\mathbf{v}^2)\,\operatorname{sgn}(f-g)(Q_\sigma(f,f)-Q_\sigma(g,g))\,d\mathbf{v}$$

$$\leqslant c\,\|f-g\|_{1/2}\,\|f+g\|_{3/2}+ch\,\|f-g\|_1\,\|f+g\|_1$$

$$+ch^2\,\|f-g\|_{1/2}\,\|f+g\|_{1/2}$$

$$\leqslant c(1+h+h^2)\,\|f-g\|_1\,\|f+g\|_{3/2}\leqslant c\,\|f-g\|_1\,\|f+g\|_{3/2}$$

for any bounded h. ∎

3. EXISTENCE AND STABILITY RESULTS

There exists an extensive literature on the Cauchy problem for the spatially homogeneous Boltzmann equation. In particular, Arkeryd's proof[1] of the existence of solutions includes all cases we are treating here. More precisely, for a collision kernel satisfying Carleman's condition (2.4), there exists a unique non-negative solution to the Cauchy problem (2.1) with a non-negative initial data $f_0 \in L_1^1$. Moreover, if $f_0 \in L_s^1$, $s > 1$, then this property is conserved in time, i.e., $f(t, \cdot) \in L_s^1$.

These results can be applied to (2.6), (2.9) and (2.10) since corresponding collision kernels have symmetry properties and satisfies Carleman's condition (2.4) (Lemma 1). As a consequence, existence, uniqueness and stability results also hold for the Cauchy problems (2.6), (2.9) and (2.10).

We could so far go directly to the next section but in view of the truncations we have introduced previously (see (2.7) and (2.8)) one can look at the problem of existence and stability in a new way. More precisely, solutions in $\mathbb{R}^d$ can be considered as limits of solutions of the truncated problems. For the sake of consistency, we present in detail a proof based on that idea, keeping in mind that many useful preliminary results are not new at all. That proof can be sketched in three steps: the first one consists in proving that there exists a unique solution to the truncated problem. Second, one shows that the sequence of solutions with increasing domains is the Cauchy sequence in a suitable Banach space and thereby converges to a certain function f. The last step amounts to pass to the limit in the truncated equations and shows that f is the unique solution of the limiting Boltzmann equation (or its discrete version).

Step 1. We refer to Arkeryd[1] to get the existence of a unique solution of (2.9) and (2.10) in a given ball $B(0, n)$.

Proposition 1. Assume that $0 \leqslant q(w, \mathbf{u}) \leqslant K$ for every $w \in \mathbb{R}^+$, $\mathbf{u} \in S^{d-1}$, where K is a given positive constant. Then there exists a unique and non-negative solution $f \in C^1(\mathbb{R}^+, L^1(\mathbb{R}^d))$ of (2.1) for every $f^0 \geqslant 0$ in L^1 and we have

$$\|f(t, \cdot)\|_0 = \|f^0\|_0, \qquad \text{for every} \quad t > 0$$

Moreover, if $f^0 \in L_s^1$ with $s \geqslant 1$, then

$$\|f(t, \cdot)\|_1 = \|f^0\|_1, \qquad \text{and} \qquad \|f(t, \cdot)\|_s \leqslant c_{t_1} \|f^0\|_s, \qquad 0 \leqslant t \leqslant t_1$$

where c_{t_1} depends only on t_1, $\|f^0\|_1$ and s but not on K.

In this proposition, the boundedness of q allows to prove the existence for a small time interval while its symmetry property allows to iterate this procedure for any time. It is therefore of major importance that truncated collision kernels (2.7) and (2.8) are bounded and preserve the collision invariance (cf. Lemma 1).

One should remark that the proof is so far not different from Arkeryd's proof, the only difference appears in the truncation: in one case the collision kernel is truncated with respect to relative velocity while in the other velocities themselves are bounded.

It is also important to remark "the mild influence of q on the moment estimates for bounded time intervals" (cf. Arkeryd[1]). In particular, moment estimations are independent on the upper value of q, i.e., on the size of the domain.

Step 2. Let us now look at the sequence $\{f^n\}_{n\in\mathbb{N}}$ (resp. $\{f^n_{\mathrm{h}}\}_{n\in\mathbb{N}}$) of solutions of the truncated problems (2.9) (resp. (2.10)) in $B(0, n)$.

Proposition 2. Let $f^0 \in L^1_s$, with $s > \frac{3}{2}$. Let f^n (resp. f^n_{h}) be a solution of (2.9) (resp. (2.10)) in $B(0, n)$ and $\{f^n\}_{n\in\mathbb{N}}$ ($\{f^n_{\mathrm{h}}\}_{n\in\mathbb{N}}$) be a sequence of solutions for increasing domains. Then $\{f^n\}_{n\in\mathbb{N}}$ ($\{f^n_{\mathrm{h}}\}_{n\in\mathbb{N}}$) converges to a non-negative function $f(f_{\mathrm{h}})$ in $C^0([0, t_1], L^1_r(\mathbb{R}^d)) \cap C^1([0, t_1], L^1_{r-1/2}(\mathbb{R}^d))$ for arbitrary t_1 and $r < s$. Moreover, $f(t, \cdot) \in L^1_s$.

Proof. We present the proof only for $\{f^n_{\mathrm{h}}\}_{n\in\mathbb{N}}$ as the continuous case is included in that case (take $h = 0$). We shall omit the index h and replace f^n_{h} by f^n, Q^n_{h} by Q^n and σ^n_{h} by σ^n. Let n, m be two integers such that $m \leqslant n$. Then using a standard argument, we have

$$\begin{aligned}
\frac{d}{dt} \|f^n - f^m\|_1 &= \int (1 + \mathbf{v}^2) \operatorname{sgn}(f^n - f^m) \left(\frac{\partial f^n}{\partial t} - \frac{\partial f^m}{\partial t}\right) d\mathbf{v} \\
&= \int (1 + \mathbf{v}^2) \operatorname{sgn}(f^n - f^m)(Q^n(f^n, f^n) - Q^m(f^m, f^m))\, d\mathbf{v} \\
&= \int (1 + \mathbf{v}^2) \operatorname{sgn}(f^n - f^m)(Q^n(f^n, f^n) - Q^n(f^m, f^m))\, d\mathbf{v} \\
&\quad + \int (1 + \mathbf{v}^2) \operatorname{sgn}(f^n - f^m)((Q^n(f^m, f^m) - Q^m(f^m, f^m))\, d\mathbf{v} \\
&\equiv E_1 + E_2
\end{aligned}$$

Applying Lemma 3 and Proposition 1, we obtain

$$E_1 \leqslant c \, \|f^n - f^m\|_1 \, \|f^n + f^m\|_{3/2} \leqslant c_{t_1} \, \|f^n - f^m\|_1 \, \|f^0\|_{3/2} \tag{3.1}$$

for arbitrary t_1 (c_{t_1} does not depend on n and m).

For the second term we have the estimation

$$E_2 \leqslant \int (1 + \mathbf{v}^2)(\sigma^n(\mathbf{V}) - \sigma^m(\mathbf{V})) \, |f'f'_1 - ff_1| \, d\mathbf{V}$$

$$\leqslant \int (1 + \mathbf{v}^2)(\sigma(\mathbf{V}) - \sigma^m(\mathbf{V})) \, |f'f'_1 - ff_1| \, d\mathbf{V}$$

as $\sigma \geqslant \sigma^n$. Here $\mathbf{V} = (\mathbf{v}', \mathbf{v}'_1, \mathbf{v}, \mathbf{v}_1)$, $f' = f^m(\mathbf{v}')$, $f'_1 = f^m(\mathbf{v}'_1)$, $f = f^m(\mathbf{v})$ and $f_1 = f^m(\mathbf{v}_1)$. Symmetry properties of σ and σ^m and non-negativeness of f^m yield

$$E_2 \leqslant \frac{1}{2} \int (2 + \mathbf{v}^2 + \mathbf{v}_1^2)(\sigma(\mathbf{V}) - \sigma^m(\mathbf{V}))(f'f'_1 + ff_1) \, d\mathbf{V}$$

Now,

$$\begin{aligned} \sigma(\mathbf{V}) - \sigma^m(\mathbf{V}) &\leqslant H(C_{\mathrm{h}}(\mathbf{v})^2 + C_{\mathrm{h}}(\mathbf{v}_1)^2 - m^2) \, \sigma(\mathbf{V}) \\ &= H(C_{\mathrm{h}}(\mathbf{v}')^2 + C_{\mathrm{h}}(\mathbf{v}'_1)^2 - m^2) \, \sigma(\mathbf{V}) \end{aligned}$$

where $H(x)$ is the Heaviside function. Using again microreversibility and bound (iii) of Lemma 1, we obtain

$$\begin{aligned} E_2 &\leqslant \int (1 + \mathbf{v}^2 + \mathbf{v}_1^2) \, H(C_{\mathrm{h}}(\mathbf{v})^2 + C_{\mathrm{h}}(\mathbf{v}_1)^2 - m^2) \, \sigma(\mathbf{V}) \, ff_1 \, d\mathbf{V} \\ &\leqslant c \int_{C_{\mathrm{h}}(\mathbf{v}) \geqslant m/\sqrt{2}} (1 + \mathbf{v}^2)^{3/2} \, f \, d\mathbf{v} \int_{\mathbb{R}^d} (1 + \mathbf{v}_1^2)^{3/2} \, f_1 \, d\mathbf{v}_1 \\ &\leqslant c_s m^{3/2 - s} \, \|f^m\|_s \, \|f^m\|_{3/2} \leqslant c_{t_1, s} m^{3/2 - s} \, \|f^0\|_s \, \|f^0\|_{3/2} \end{aligned} \tag{3.2}$$

for any $s > \frac{3}{2}$ and bounded h. This last inequality together with (3.1) gives

$$\frac{d}{dt} \|f^n - f^m\|_1 \leqslant c_{t_1, s, f^0}(\|f^n - f^m\|_1 + m^{3/2 - s}) \tag{3.3}$$

Since

$$\|f^n(0,\cdot)-f^m(0,\cdot)\|_1 \leqslant \int_{C_h(\mathbf{v})\geqslant m} (1+\mathbf{v}^2)\, f^0(\mathbf{v})\, d\mathbf{v} \leqslant cm^{1-s}\, \|f^0\|_s,$$

then using Gronwall's lemma, we obtain

$$\|f^n(t,\cdot)-f^m(t,\cdot)\|_1 \leqslant Cm^{3/2-s}e^{Ct}, \qquad \text{for} \quad 0 \leqslant t \leqslant t_1$$

Hence, $\{f^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence of non-negative functions which converges to a non-negative function f in $C^0([0,t_1], L_1^1(\mathbb{R}^d))$. Moreover, using Fatou's lemma, one can pass to the limit in

$$\|f^n(t,\cdot)\|_s \leqslant c_{t_1}\, \|f^0\|_s, \qquad 0 \leqslant t \leqslant t_1$$

so that $\|f(t,\cdot)\|_s \leqslant c_{t_1,s,f^0}$. Hence,

$$\begin{aligned}\|f^n(t,\cdot)-f(t,\cdot)\|_r &\leqslant \int_{|\mathbf{v}|\leqslant R} + \int_{|\mathbf{v}|>R}\\ &\leqslant (1+R^2)^{r-1}\, \|f^n(t,\cdot)-f(t,\cdot)\|_1 + 2c_{t_1,s,f^0}(1+R^2)^{r-s}\end{aligned}$$

for $0 \leqslant t \leqslant t_1$ and arbitrary R. A suitable choice of R and n yields

$$f^n \to f \qquad \text{in} \quad C^0([0,t_1], L_r^1(\mathbb{R}^d)), \quad r<s \tag{3.4}$$

Now, Lemma 2 gives

$$\left\|\frac{\partial f^n}{\partial t}-\frac{\partial f^m}{\partial t}\right\|_{r-1/2} \leqslant c\, \|f^n-f^m\|_r\, \|f^n+f^m\|_r \leqslant c_{t_1,s,f^0}\, \|f^n-f^m\|_r$$

Therefore $\{f^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^1([0,t_1], L^1_{r-1/2}(\mathbb{R}^d))$, $r<s$, which converges to a function F. But according to (3.4), that limit must be f so that $F=f$ a.e. ∎

Step 3. We can now give a proof of existence, uniqueness and stability for the Cauchy problems (2.1) and (2.6) (we observe that (2.5) and (2.6) are in fact two different formulations of the same problem).

Theorem 1. Let f^0 be a non-negative function in L_s^1, with $s>\frac{3}{2}$. Then, for an arbitrary t_1 there exists a unique and non-negative solution f to the Cauchy problem (2.1), $f \in C^0([0,t_1], L_r^1(\mathbb{R}^d)) \cap C^1([0,t_1],$

$L^1_{r-1/2}(\mathbb{R}^d)$), $r < s$. This solution is a uniform limit of solutions to the truncated problems as described in Proposition 2. Moreover, we have $\|f(t,\cdot)\|_0 = \|f^0\|_0$, $\|f(t,\cdot)\|_1 = \|f^0\|_1$ and

$$\|f(t,\cdot)\|_s \leqslant c_{t_1} \|f^0\|_s, \qquad t \leqslant t_1 \tag{3.5}$$

where c_{t_1} depends only on t_1 and moments of f^0.

The same conclusion is valid for the discrete problem (2.6) (or equivalently (2.5)).

Proof. Let f be the limit of a sequence $\{f^n\}_{n\in\mathbb{N}}$ of solutions in bounded domains (see Proposition 2). Then $f(t,\cdot) \in L^1_r(\mathbb{R}^d)$ and Lemma 2 implies that $Q(f,f)(t,\cdot) \in L^1_{r-1/2}(\mathbb{R}^d)$ $(r \leqslant s,\ t \leqslant t_1)$. After calculations similar to Proposition 2 one obtains

$$\begin{aligned}
&\|Q^n(f^n, f^n)(t,\cdot) - Q(f,f)(t,\cdot)\|_0 \\
&\quad \leqslant c\,\|f^n - f\|_{1/2}\,(\|f^n\|_{1/2} + \|f\|_{1/2}) + cn^{1/2-s}\,\|f\|_s\,\|f\|_{1/2} \\
&\quad \leqslant c_{t_1, f^0}\Big(\sup_{0\leqslant t\leqslant t_1} \|f^n - f\|_{1/2} + n^{1/2-s}\Big) \to 0, \qquad \text{as} \quad n \to \infty
\end{aligned}$$

where we have used the uniform boundedness of moments of f^n and f. Hence, $Q^n(f^n, f^n) \to Q(f,f)$ strongly in $C^0([0,t_1], L^1(\mathbb{R}^d))$.

On the other hand, $\partial f^n/\partial t \to \partial f/\partial t$ in $C^0([0,t_1], l^1_{r-1/2}(\mathbb{R}^d))$, $r < s$, so that f is a solution of the Boltzmann equation in the whole domain.

If g is also a solution of the Boltzmann equation with the same initial data, Lemma 3 gives

$$\frac{d}{dt}\|f-g\|_1 \leqslant c_{t_1, f^0}\,\|f-g\|_1$$

for $0 \leqslant t \leqslant t_1$. Applying Gronwall's lemma, we obtain $f = g$. ∎

Remark 2. Results on existence and uniqueness of solutions for the continuous Boltzmann equation better than Theorem 1 are known in the literature for a long time. The most recent is the theorem by Mischler and Wennberg[14] in which it is assumed only that $f^0 \in L^1_1$. The advantage of our Theorem 1 is the fact that the theorem remains valid both in continuous and discrete case. Essential for our future considerations is, however, the fact that solutions of the truncated problems converge to the solution of the Cauchy problem in $\mathbb{R}^d$.

4. CONVERGENCE

We can now study the convergence of discrete solutions to the continuous solution of the space homogeneous Boltzmann equation (2.1). The convergence leans essentially on the approximation (2.5) of the collision integral. The following consistency result is taken from our previous paper.[15]

Proposition 3. Let q be a continuous collision kernel such that (2.4) holds and f be a function in $C^0(\mathbb{R}^d) \cap L^1_s(\mathbb{R}^d)$, with $s > 2$, $d \geqslant 3$. Then

$$\sum_{j,k,l \in I_i} \Gamma_{ij}^{kl}(f_k f_l - f_i f_j) \to Q(f, f)(\mathbf{v}_i) \qquad \text{as} \quad h \to 0, \tag{4.1}$$

uniformly on every compact set T.

That result has to be understood as follows: suppose that $\mathbf{v}$ is a given velocity in a compact set T and denote $\Omega_h(\mathbf{v}) = \{\mathbf{w} \in \mathbb{R}^d : \mathbf{w} = \mathbf{v} + h\mathbf{n}, \mathbf{n} \in \mathbb{Z}^d\}$, then the previous quadrature applied in $\mathbf{v}$ with quadrature points in $\Omega_h(\mathbf{v})$ converges to $Q(f, f)(\mathbf{v})$ as $h \to 0$. Moreover, this convergence is uniform with respect to $\mathbf{v}$ in T.

We are concerned with the approximation of solution of the Boltzmann equation by numerical solutions. From that point of view, discrete velocity models with infinite number of velocities cannot be considered and we have to use truncated discrete velocity models which are well suited for numerical purposes. Then we expect to obtain a result of the form: solutions of truncated discrete velocity models tend to the continuous solution as the domain of computation is increased and the step h diminished. This is the conclusion of the following theorem.

Theorem 2. Let f be a solution of the space homogeneous Boltzmann equation (2.1) with initial data $f^0 \in C^0(\mathbb{R}^d) \cap L^1_s(\mathbb{R}^d)$, for $s > \frac{3}{2}$, $d \geqslant 3$. Let f^n_{h} be a solution of the Cauchy problem for DVM (2.9) in the ball $B(0, n)$ with step h and initial data $f^0(C_{\mathrm{h}}(\mathbf{v}))$. Then

$$\|f(t) - f^n_{\mathrm{h}}(t)\|_1 \leqslant c(\varepsilon(h) + n^{3/2-s})\, e^{ct}, \qquad \text{for} \quad t \in [0, t_1]$$

where $\varepsilon(h) \to 0$ as $h \to 0$ and c is a constant depending only on t_1, f^0 and s.

Proof. From the assumptions of the theorem and results of Carleman[5] and Arkeryd[1] (see Proposition 1) it follows that the solution of Eq. (2.1) is in $C^0(\mathbb{R}^d) \cap L^1_s(\mathbb{R}^d)$. Analogously, from Theorem 1, $f^n_{\mathrm{h}}(t)$ is in L^1_s with the norm independent of n. Hence we can consider the norm of the difference of these two solutions

$$\begin{aligned}\frac{d}{dt}\|f-f^n_{\rm h}\|_1 &= \int (1+\mathbf{v}^2)(Q(f,f)-Q^n_{\rm h}(f^n_{\rm h},f^n_{\rm h}))\,\text{sgn}(f-f^n_{\rm h})\,d\mathbf{v}\\ &= \int (1+\mathbf{v}^2)((Q(f,f)-Q_{\rm h}(f,f))+(Q_{\rm h}(f,f)-Q^n_{\rm h}(f,f))\\ &\quad +(Q^n_{\rm h}(f,f)-Q^n_{\rm h}(f^n_{\rm h},f^n_{\rm h})))\,\text{sgn}(f-f^n_{\rm h})\,d\mathbf{v}\\ &\equiv E_1+E_2+E_3\end{aligned}$$

We shall now estimate the three terms separately. For the first term we have

$$E_1 \leqslant \int (1+\mathbf{v}^2)\,|Q(f,f)-Q_{\rm h}(f,f)|\,d\mathbf{v} = \int_{|\mathbf{v}|\leqslant m} + \int_{|\mathbf{v}|>m} \equiv E_{11}+E_{12}$$

For $|\mathbf{v}|\leqslant m$ we make the following decomposition

$$\begin{aligned}|Q(f,f)(\mathbf{v})-Q_{\rm h}(f,f)(\mathbf{v})| \leqslant{}& |Q(f,f)(\mathbf{v})-Q(f,f)(C_{\rm h}(\mathbf{v}))|\\ &+|Q(f,f)(C_{\rm h}(\mathbf{v}))-Q_{\rm h}(f,f)(C_{\rm h}(\mathbf{v}))|\\ &+|Q_{\rm h}(f,f)(C_{\rm h}(\mathbf{v}))-Q_{\rm h}(f,f)(\mathbf{v})|\end{aligned}$$

$Q(f,f)$ is uniformly continuous on compact set $[0,t_1]\times B(0,m)$, so that the first term tends uniformly to zero on that set. Second term is the error of the quadrature formula (4.1) which tends uniformly to zero on $[0,t_1]\times B(0,m)$. As concerns the last term, we remark that the positive part of $Q_{\rm h}(f,f)$ is piecewise constant so that

$$\begin{aligned}&|Q_{\rm h}(f,f)(\mathbf{v})-Q_{\rm h}(f,f)(C_{\rm h}(\mathbf{v}))|\\ &\quad\leqslant |f(\mathbf{v})-f(C_{\rm h}(\mathbf{v}))|\int_{\mathbb{R}^3} f(\mathbf{v}_1)\,\sigma_{\rm h}(\mathbf{v},\mathbf{v}_1,\mathbf{v}',\mathbf{v}'_1)\,d\mathbf{v}_1\,d\mathbf{v}'\,d\mathbf{v}'_1\\ &\quad\leqslant c_{t_1}(1+\mathbf{v}^2)^{1/2}\,|f(\mathbf{v})-f(C_{\rm h}(\mathbf{v}))|\,\|f^0\|_{1/2}\end{aligned}$$

and consequently that term converges uniformly to zero on $[0,t_1]\times B(0,m)$. Finally, we obtain

$$E_{11}\leqslant c_{t_1,f^0}\,\varepsilon_m(h)$$

where $\varepsilon_m(h)$ tends to zero with h for fixed m.

For $|\mathbf{v}| > m$ we can write

$$E_{12} \leqslant c \int_{|\mathbf{v}| > m} \int_{\mathbb{R}^d} (1 + \mathbf{v}^2)^{3/2} (1 + \mathbf{v}_1^2)^{1/2} ff_1 \, d\mathbf{v} \, d\mathbf{v}_1$$

$$+ \int_{|\mathbf{v}| > m} \int_{\mathbb{R}^{3d}} (1 + \mathbf{v}^2)(\sigma(\mathbf{V}) + \sigma_{\mathrm{h}}(\mathbf{V})) \, f'f'_1 \, d\mathbf{V}$$

The first term is easy to bound, while for the second we have to use inequalities (2.12) and

$$m^2 \leqslant \mathbf{v}^2 + \mathbf{v}_1^2 \leqslant c(|\mathbf{v}'| + |\mathbf{v}'_1| + h)^2$$

Proceeding as in (3.2) we obtain

$$E_{12} \leqslant c_{t_1, s} m^{3/2 - s} \, \|f^0\|_s \, \|f^2\|_{3/2}$$

Therefore

$$E_1 \leqslant c_{t_1, f^0, s}(\varepsilon_m(h) + m^{3/2 - s})$$

for any $m \in \mathbb{R}^+$. For ε arbitrary small, let m be such that $\varepsilon/2 = m^{3/2 - s}$ and h such that $\varepsilon_m(h) \leqslant \varepsilon/2$. Then

$$E_1 \leqslant c_{t_1, f^0, s} \, \varepsilon$$

where $\varepsilon \to 0$ as $h \to 0$.

We can now proceed to the estimation of E_2 which is actually similar to (3.2) (see Proposition 2). Then

$$E_2 \leqslant c_{t_1, f^0, s} n^{3/2 - s}$$

Finally one can apply Lemma 3 to bound E_3

$$E_3 = \int (1 + \mathbf{v}^2)(Q^n_{\mathrm{h}}(f, f) - Q^n_{\mathrm{h}}(f^n_{\mathrm{h}}, f^n_{\mathrm{h}})) \, \mathrm{sgn}(f - f^n_{\mathrm{h}}) \, d\mathbf{v}$$

$$\leqslant c \, \|f - f^n_{\mathrm{h}}\|_1 \, \|f + f^n_{\mathrm{h}}\|_{3/2} \leqslant c_{t_1, f^0} \, \|f - f^n_{\mathrm{h}}\|_1$$

Collecting the estimates for E_1, E_2 and E_3, we have

$$\frac{d}{dt} \|f - f^n_{\mathrm{h}}\|_1 \leqslant c_{t_1, f^0, s}(\varepsilon(h) + n^{3 - 2s} + \|f - f^n_{\mathrm{h}}\|_1),$$

which using Gronwall's lemma gives the assertion of the theorem. ∎

5. THE SPACE DEPENDENT BOLTZMANN EQUATION

In this section we consider the Cauchy problem for the space dependent Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = Q(f, f), \qquad \mathbf{v} \in \mathbb{R}^d, \quad \mathbf{x} \in \mathbb{R}^d$$
$$f(0) = f^0 \tag{5.1}$$

This problem have been considered by Mischler[13] in the frame of the DiPerna–Lions solutions.[7] To obtain the renormalized solution of (5.1) in the sense of DiPerna–Lions it is necessary to assume that the initial data f^0 fulfills the condition

$$\int f^0(\mathbf{x}, \mathbf{v})(1 + \mathbf{x}^2 + \mathbf{v}^2 + |\log f^0(\mathbf{x}, \mathbf{v})|)\, d\mathbf{x}\, d\mathbf{v} \leqslant c \tag{5.2}$$

Then the Cauchy problem (5.1) possesses a renormalized solution $f(t)$ which satisfies the condition

$$\sup_{[0, t_1]} \int f(t, \mathbf{x}, \mathbf{v})(1 + \mathbf{x}^2 + \mathbf{v}^2 + |\log f(t, \mathbf{x}, \mathbf{v})|\, d\mathbf{x}\, d\mathbf{v} \leqslant c_{t_1}$$

To approximate this solution by a discrete velocity model we discretize the velocity space like for the space-homogeneous equation and leave the configuration space unchanged. This leads to the hyperbolic system of equations

$$\frac{\partial f_i}{\partial t} + C_{\mathbf{h}}(\mathbf{v}) \cdot \nabla_{\mathbf{x}} f_i = \sum_{j, k, l \in I_i} \Gamma_{ij}^{kl}(f_k f_l - f_i f_j)$$
$$f_i(0) = f_i^0(\mathbf{x}) \tag{5.3}$$

As in the previous section, we have to show an existence theorem for that system. Because there is no global existence result for DVM, Mischler[13] has modified this system of equations by a truncation

$$\frac{\partial f_i}{\partial t} + C_{\mathbf{h}}(\mathbf{v}) \cdot \nabla_{\mathbf{x}} f_i = \kappa_R(\rho) \sum_{j, k, l \in I_i} \Gamma_{ij}^{kl}(f_k f_l - f_i f_j)$$
$$f_i(0) = f_i^0(\mathbf{x}) \tag{5.4}$$

where $\rho = \sum h^d f_i$ and $\kappa_R(x) = \min(x, R)/x$.

The last equation can be written in terms of the step function f_h like in the previous section. This time, however, $f_h = f_h(t, \mathbf{x})$. Then we have

$$\frac{\partial f_h}{\partial t} + C_h(\mathbf{v}) \cdot \nabla_{\mathbf{x}} f_h = \kappa_{R_h}(\rho_h)\, Q_h(f_h, f_h)$$
$$f_h(0) = f_h^0 \tag{5.5}$$

where $\rho_h = \int_{\mathbb{R}^d} f_h \, d\mathbf{v}$ and R_h is a positive constant, which will be chosen later. Then, as Mischler observed, the global existence theorem follows by the Banach fixed point theorem.

The proof of convergence of solution to the system (5.3) (or (5.5)) is due to Mischler.[13] However, as the author observed himself, there is a missing gap in the proof. To be more precise, Mischler proves convergence under very strong assumption about the approximation of the collision integral by a discrete sum. Lemma 4 below proves that this assumption is fulfilled for our class of approximate collision kernels σ_h.

Lemma 4. For every test function $\phi \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and almost all $\mathbf{v}, \mathbf{v}_1$ in $\mathbb{R}^d$, $d \geqslant 3$,

$$\int_{\mathbb{R}^{2d}} \sigma_h(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}_1')\, \phi(\mathbf{v}')\, d\mathbf{v}'\, d\mathbf{v}_1' \to \int_{S^{d-1}} q(|\mathbf{v} - \mathbf{v}_1|, \mathbf{u})\, \phi(\mathbf{v}')\, d\mathbf{u} \tag{5.6}$$

as $h \to 0$.

Proof. According to the definition of σ_h the integral in the left hand side of (5.6) is in fact a sum which is defined as follows. We take two vectors $\mathbf{v}_i = C_h(\mathbf{v})$ and $\mathbf{v}_j = C_h(\mathbf{v}_1)$ and span the sphere $S^{d-1}(\mathbf{v}_i, \mathbf{v}_j)$ with the center $(\mathbf{v}_i + \mathbf{v}_j)/2$ and diameter $|\mathbf{v}_i - \mathbf{v}_j|$. Then we consider all points from the grid Ω_h which belong to that sphere and form pairs of antipodal points. Denoting by $(\mathbf{v}_k, \mathbf{v}_l)$ such pairs and by r_{ij} their number, we can write

$$\int_{\mathbb{R}^{2d}} \sigma_h(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}_1')\, \phi(\mathbf{v}')\, d\mathbf{v}'\, d\mathbf{v}_1' = \frac{1}{r_{ij}} \sum_{k,l} q\left(|\mathbf{v}_i - \mathbf{v}_j|, \frac{\mathbf{v}_k - \mathbf{v}_l}{|\mathbf{v}_k - \mathbf{v}_l|}\right) \phi(\mathbf{v}_k)$$

Then it is clear that the statement of the lemma is exactly Corollary 4 from our paper.[15] The only difference is that the corollary has been proved for $d = 3$ but the proof is similar for $d \geqslant 3$. ∎

Taking in (5.5) a sequence of truncation constants R_h such that $R_h \to \infty$ as $h \to 0$, we obtain the following theorem on convergence. Its proof is a combination of the original proof due to Mischler[13] and our Lemma 4.

Theorem 3. For $h \to 0$ and every $t_1 > 0$ a sequence of solutions f_h of the initial-value problem (5.5) with initial data fulfilling condition (5.2) converges weakly in $L^\infty([0, t_1], L^1(\mathbb{R}^{2d}))$, $d \geqslant 3$, up to the extraction of a subsequence, to a renormalized solution of the Boltzmann Eq. (5.1).

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